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LETTER TO THE EDITOR

Integrity bases for symmetry-conserving higher-order interaction terms in the interacting boson model

Joris Van der Jeugt†

Seminarie voor Wiskundige Natuurkunde, Rijksuniversiteit-Gent, Krijgslaan 281-S9, B9000 Gent, Belgium

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Abstract. Using generating function techniques, we construct integrity bases for O(3) scalar operators in the enveloping algebra of G, where G is one of the subgroups of SU(6) in the three dynamical chains for the interacting boson model.

In the original interacting boson model (IBM), initially introduced by Arima and Iachello (1976, 1978, 1979), a dynamical symmetry arises whenever the Hamiltonian H can be written in terms of invariants only of maximal subgroups $G \subset SU(6)$. In these papers, the Hamiltonian was an expression up to second order in the SU(6) generators (i.e. only two-boson interaction terms were considered). Recently, Vanden Berghe *et al* (1985) introduced higher-order terms which conserve the dynamical symmetry in the Hamiltonian. Such terms must be operators belonging to the integrity basis of O(3) scalar operators in the enveloping algebra of G ($G = SU(5)$, SU(3) or O(6)). For the SU(3) symmetry, this integrity basis is determined by the generating function (Judd *et al* 1974, Gilmore and Draayer 1985)

$$(1+U^6)/(1-U^3)(1-U^4). \tag{1}$$

This generating function (GF) shows that, besides the Casimir operators of SU(3) and the O(3) Casimir operator L^2 (for which the corresponding denominators are already left out of (1)), there are still two functionally independent operators of third and fourth degree in the SU(3) generators. Moreover, there is an operator of degree six whose square can be expressed in terms of these and of Casimir operators. The functionally independent operators, sometimes denoted by Ω and Λ, were added to the Hamiltonian in the SU(3) dynamical chain (Vanden Berghe *et al* 1985) and gave rise to a much better approximation of the energy spectrum. Also, they removed the degeneracy which exists for members of the β and γ bands by using the original Hamiltonian.

For the other two dynamical symmetries, namely $SU(6) \supset O(6) \supset O(5) \supset O(3)$ and $SU(6) \supset SU(5) \supset O(5) \supset O(3)$, the relevant integrity bases of O(3) scalars are so far unknown. However, if one wishes to preserve only the O(5) symmetry in both chains, the corresponding GF can be found in Giroux *et al* (1984, equation (3.7)):

$$(1+U^9)/(1-U^4)(1-U^6). \tag{2}$$

† Senior Research Assistant NFWO (Belgium).

Hence, acting on states of symmetric O(5) representations, there are two functionally independent O(3) scalars in the O(5) enveloping algebra, namely one of fourth and one of sixth order. In the notation of Iachello (1980), the fourth-order operator can be constructed by coupling the $G_\mu^{(3)}(dd)$ tensor together with three $G_\upsilon^{(1)}(dd)$ tensors to a tensor of rank 0.

But if we want to preserve the whole dynamical symmetry in the two chains, the integrity basis for O(3) scalars in the enveloping algebra of O(6) and of SU(5) needs to be constructed. First, consider the O(6) chain. Since O(6) ~ SU(4) and the symmetric O(6) irreducible representations (irreps) $(\sigma, 0, 0)$ correspond to the representations $(0, \sigma, 0)$ of SU(4), we find from Giroux *et al* (1984, equation (4.14)) that the generating function for λ tensors in the degenerate enveloping algebra is

$$[(1-U^2\Lambda_2^2)(1-U\Lambda_1\Lambda_3)]^{-1}. \tag{3}$$

Here, Λ_i carries the Dynkin label λ_i of SU(4) irreps as exponent. Now, we have to find the GF for O(3) scalars contained in SU(4) irreps of the form (v, w, v). This is performed in two steps. First, we have constructed the branching rule GF for SU(4) irreps (v, w, v) into O(5) irreps (p, q):

$$[(1-VQ)(1-VP^2)(1-W)(1-WQ)]^{-1}. \tag{4}$$

Then the GF for O(3) scalars contained in O(5) irreps (p, q) is given by equation (3.5) of Giroux *et al* (1984):

$$(1+P^6Q^3)[(1-P^4)(1-Q^3)(1-P^4Q^2)]^{-1}. \tag{5}$$

Replacing (P, Q) by (P^{-1}, Q^{-1}) in (4), multiplying by (5) and retaining the coefficient in P^0Q^0 gives

$$\begin{aligned}[V^2W^2/(1-V^2)(1-VW)+(1+V^2W+VW^2)/(1-V^3)\\ +(V^4+V^5W^2+V^3W)/(1-V^2)(1-V^3)][(1-W)(1-V^2)(1-W^3)]^{-1}.\end{aligned} \tag{6}$$

This is the GF for the number of O(3) scalars contained in an SU(4) irrep (v, w, v). Now, according to (3), we only have to multiply (6) by $[(1-U^2W^{-2})(1-UV^{-1})]^{-1}$ and retain the part in V^0W^0. Omitting a factor $(1-U^2)^{-1}$ corresponding to the O(3) second degree Casimir, this gives rise to

$$\frac{1+3U^4+U^5+3U^6+U^{10}}{(1-U^2)(1-U^3)^2(1-U^4)}. \tag{7}$$

Hence, disregarding the O(6) and O(3) Casimir operators, there are four functionally independent O(3) scalar operators in the enveloping algebra of O(6) for symmetric $(\sigma, 0, 0)$ irreps: one of second degree, two of third degree and one of fourth degree. The second degree operator is the O(5) Casimir. With $l_\mu = G_\mu^{(1)}(dd)$, $p_\mu = G_\mu^{(2)}(ds)+G_\mu^{(2)}(sd)$ and $q_\mu = G_\mu^{(3)}(dd)$, the third degree operators can be written as $[p\times[l\times l]^{(2)}]^{(0)}$ and $[[p\times q]^{(1)}\times l]^{(0)}$. The operator of degree four is the O(3) scalar in O(5), namely $[q\times[l\times l\times l]^{(3)}]^{(0)}$. Besides these four functionally independent operators, the integrity bases of O(3) scalars in the degenerate enveloping algebra of

O(6) still contain three other fourth degree operators, one fifth degree operator and three operators of degree six.

Finally, we shall construct the GF for the O(3) scalars in the degenerate enveloping algebra of SU(5), i.e. when acting on states of symmetric irreps $(n_d, 0, 0, 0)$ of SU(5). First, the GF for λ tensors in the degenerate enveloping algebra is given by (King *et al* 1982)

$$(1 - U\Lambda_1\Lambda_4)^{-1}. \tag{8}$$

Hence, we have to find the GF for O(3) scalars contained in SU(5) irreps of the form $(v, 0, 0, v)$. Just as in the previous situation, we first construct the branching rule GF for SU(5) irreps $(v, 0, 0, v)$ into O(5) irreps (p, q):

$$\frac{1 + V^2Q^2}{(1 - VP^2)(1 - VQ^2)(1 - V^2)(1 - V^2Q^2)}. \tag{9}$$

For O(5) irreps (p, q), the O(3) scalar content is given by (5). Replacing (P, Q) by (P^{-1}, Q^{-1}) in (9), multiplying by (5) and retaining the coefficient in P^0Q^0 gives the GF for the number of O(3) scalars contained in an SU(5) irrep $(v, 0, 0, v)$. Then, according to (8), we need only need to multiply this latter expression by $(1 - UV^{-1})^{-1}$ and retain the part in V^0. Omitting a factor $(1 - U^2)^{-1}$ corresponding to the O(3) second degree Casimir, one obtains

$$\frac{1 + 3U^4 + 2U^5 + 3U^6 + U^{10}}{(1 - U^2)(1 - U^3)^2(1 - U^4)}. \tag{10}$$

This expression is very close to the GF (7). In fact, it shows that the integrity basis of O(3) scalars in the degenerate enveloping algebra of SU(5) contains four functionally independent scalars, three operators of fourth degree, two operators of fifth degree and three operators of degree six. The four functionally independent operators have the same degrees as in the case of O(6). The second degree operator is again the O(5) Casimir. With $T^{(k)}_\kappa = G^{(k)}_\kappa(dd)$ $(k = 1, 2, 3, 4)$, the third degree operators can be taken as $[T^{(2)} \times [T^{(1)} \times T^{(1)}]^{(2)}]^{(0)}$ and $[[T^{(3)} \times T^{(2)}]^{(1)} \times T^{(1)}]^{(0)}$. It follows from (10) that any other third degree O(3) scalar, for instance $[[T^{(4)} \times T^{(3)}]^{(1)} \times T^{(1)}]^{(0)}$, is indistinguishable from a linear combination of the first two when acting on states of totally symmetric representations $(n_d, 0, 0, 0)$. Finally, the set of functionally independent operators also contain one fourth degree operator, for which one can take the previously mentioned O(3) scalar in O(5).

The generating functions (7) and (10) solve the question of symmetry-conserving higher-order interaction terms in the O(6) and SU(5) limit of IBM completely. The explicit introduction of such higher-order terms in the Hamiltonain and physical applications will be published elsewhere.

References

Arima A and Iachello F 1976 *Ann. Phys., NY* **99** 253-317
—— 1978 *Ann. Phys., NY* **111** 201-38
—— 1979 *Ann. Phys., NY* **123** 468-92
Gilmore R and Draayer J P 1985 *J. Math. Phys.* **26** 3053-67
Giroux Y, Couture M and Sharp R T 1984 *J. Phys. A: Math. Gen.* **17** 715-25

Iachello F 1980 *Nuclear Structure* ed K Abrahams, K Allaart and A E L Dieperink (New York: Plenum) pp 53-89
Judd B R, Miller W Jr, Patera J and Winternitz P 1974 *J. Math. Phys.* **15** 1787-99
King R C, Patera J and Sharp R T 1982 *J. Phys. A: Math. Gen.* **15** 1143-58
Vanden Berghe G, De Meyer H and Van Isacker P 1985 *Phys. Rev.* C **32** 1049-56